Note on the frequency of vanishing of L-functions of elliptic curves in a family of quadratic twists

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Abstract

In this note, we give an example of an elliptic curve E such that for all prime discriminants $d < 0$ for which the sign of the functional equation of the L-function of the quadratic twist E_d of E by d is $+1$, we have $L(E_d, 1) \neq 0$. Furthermore, using the Birch and Swinnerton-Dyer conjecture, we prove that the Tate-Shafarevitch group of E_d, for all such d, has a trivial 2-part. Our method can be generalized to other examples.

1 Notations and introduction

Let E be an elliptic curve defined over $\mathbb{Q}$ with conductor N and with L-function $L(E,s) = \sum_n a(n)n^{-s}$. From the work of Wiles, Taylor ([Wil], [Tay-Wil]) and Breuil, Conrad, Diamond, Taylor ([Bre et al.]), $L(E,s)$ can be continued to the whole complex plane and satisfies a functional equation:

$$\Lambda(E,s) = \varepsilon(E)\Lambda(E, 2-s),$$

where $\varepsilon(E) = \pm 1$ is the sign of the functional equation and $\Lambda(E,s)$ is given by:

$$\Lambda(E,s) = \left(\frac{\sqrt{N(E)}}{2\pi}\right)^s \Gamma(s)L(E,s).$$

Let d be a fundamental discriminant, $\left(\frac{d}{\cdot}\right)$ its associated quadratic character and E_d the quadratic twist of E by d. Furthermore, we assume that d is prime to N. Hence the conductor of E_d is Nd^2, and the sign of the functional equation of $L(E_d, s) = \sum_n a(n)\left(\frac{d}{n}\right)n^{-s}$ is

$$\varepsilon(E_d) = \varepsilon(E)\left(\frac{d}{-N(E)}\right).$$

Classical questions are concerned with the distribution of the special values $L(E_d, 1)$ as d runs through discriminants with $\varepsilon(E_d) = 1$. For example, one can ask for the density of those d such that $L(E_d, 1) = 0$ or for the density of those d such that $p \mid |Ш(E_d)|$, where p is a fixed prime and $Ш(E_d)$ is the Tate-Shafarevitch group[1] of E_d. The Birch and Swinnerton-Dyer conjecture gives a precise link between the value $L(E_d, 1)$ and the order $|Ш(E_d)|$ of the Tate-Shafarevitch group.

Using elementary arithmetic on quadratic forms, we prove that if E is the elliptic curve "17a1" in Cremona's table ([Cre]) then for all prime discriminants $d < 0$ with $\varepsilon(E_d) = 1$, we have $L(E_d, 1) \neq 0$ and (using the Birch and Swinnerton-Dyer conjecture) $2 \nmid |Ш(E_d)|$. Other examples can be handled with the same method.

2 The example

Throughout this section E denotes the elliptic curve with conductor $N = 17$ defined by:

$$E \ : \ y^2 + xy + y = x^3 - x^2 - x - 14.$$

We consider the quadratic twists E_d of E by discriminants $d < 0$ coprime with N such that $\varepsilon(E_d) = 1$ (i.e. $d \equiv 1, 2, 4, 8, 9, 13, 15, 16 \mod 17$). By a theorem of Waldspurger ([Wal]), the values of $L(E_d, 1)$ are related to the coefficients $c(n)$ of a weight 3/2 modular form. More precisely, we have:

$$L(E_d, 1) = \frac{\kappa}{\sqrt{|d|}} c(|d|)^2, \tag{2.1}$$

where κ is a constant (here $\kappa \approx 2.74573911$) and where the modular form of weight 3/2, computed by Tornaria ([Tor]), is given by:

$$\sum_n c(n) q^n = \frac{\theta_1(q) - \theta_2(q)}{2}$$

with:

$$\begin{aligned} \theta_1(q) &= \sum_{(x,y,z)\in\mathbb{Z}^3} q^{3x^2+23y^2+23z^2-2xy-2xz-22yz} \\ \theta_2(q) &= \sum_{(x,y,z)\in\mathbb{Z}^3} q^{7x^2+11y^2+20z^2-6xy-4xz-8yz}. \end{aligned}$$

We have:

Theorem 1. *If $d < 0$ is a prime discriminant with $\varepsilon(E_d) = 1$, the coefficient $c(-d)$ is odd.*

[1] The Tate-Shafarevitch group of an elliptic curve E is some "cumbersome" group which, roughly speaking, measures the obstruction of a certain "local-global" principle (see [Sil] for a precise definition). It is conjectured that it is a finite group and, if so, one can prove that its order is a perfect square.

Proof Let d be such a discriminant and $p = -d$. Remark that we have $p \equiv 3 \mod 4$ and that $\left(\frac{-17}{p}\right) = 1$. We let:

$$\begin{aligned} Q_1(x, y, z) &= 3x^2 + 23y^2 + 23z^2 - 2xy - 2xz - 22yz \\ Q_2(x, y, z) &= 7x^2 + 11y^2 + 20z^2 - 6xy - 4xz - 8yz. \end{aligned}$$

We consider the following sets:

$$\begin{aligned} R_1 &= \{(x, y, z) \in \mathbb{Z}^3,\ Q_1(x, y, z) = p\} \\ R_2 &= \{(x, y, z) \in \mathbb{Z}^3,\ Q_2(x, y, z) = p\} \end{aligned}$$

and we must prove that $|R_1| - |R_2| \equiv 2 \mod 4$. In fact, the two ternary quadratic forms Q_1 and Q_2 are invariant by the involutions $\iota : (x, y, z) \mapsto (-x, -y, -z)$ and $\tau : (x, y, z) \mapsto (x-z, y-z, -z)$. Hence, if $P = (x, y, z) \in R_i$ (for $i = 1, 2$), then $P, \iota(P), \tau(P)$ and $\iota \circ \tau(P)$ also belong to R_i and these 4 points are distinct except if $z = 0$. Thus,

$$|R_i| \equiv |\{(x, y) \in \mathbb{Z}^2, Q_i(x, y, 0) = p\}| \mod 4.$$

Hence for $|R_1| \mod 4$, we are led to study the number of solutions of

$$p = 3x^2 - 2xy + 23y^2 = q_1(x, y)$$

and for $|R_2| \mod 4$ the number of solutions of

$$p = 7x^2 - 6xy + 11y^2 = q_2(x, y).$$

There are 8 classes of primitive quadratic forms with discriminant $\Delta = -2^4 \times 17$ modulo $SL_2(\mathbb{Z})$. As a set of representatives we can choose the 8 following ones:

$$\begin{array}{ll} q_1(x, y) = 3x^2 - 2xy + 23y^2 & \overline{q_1}(x, y) = 3x^2 + 2xy + 23y^2 \\ q_2(x, y) = 7x^2 - 6xy + 11y^2 & \overline{q_2}(x, y) = 7x^2 + 6xy + 11y^2 \\ q_3(x, y) = 8x^2 - 4xy + 9y^2 & \overline{q_3}(x, y) = 8x^2 + 4xy + 9y^2 \\ q_4(x, y) = 4x^2 + 17y^2 & q_5(x, y) = x^2 + 68y^2 \end{array}$$

Since we have $\left(\frac{-17}{p}\right) = 1$, the prime p must be represented by one of these forms. Since $p \equiv 3 \mod 4$, the prime p cannot be represented by the forms q_3, $\overline{q_3}$, q_4 and q_5. Hence, we have two possibilities:

- The prime p is represented by q_1 with only 2 solutions (and so it is for $\overline{q_1}$) and p is not represented by q_2 (neither by $\overline{q_2}$).
- The prime p is not represented by q_1 hence it is by q_2 with only 2 solutions (and so it is by $\overline{q_2}$).

In each case, we conclude that $|R_1| - |R_2| \equiv 2 \mod 4$ and so $c(p)$ is odd. □

Corollary 2. *Let $d < 0$ be a prime discriminant such that $\varepsilon(E_d) = 1$, then we have $L(E_d, 1) \neq 0$.*

Proof Indeed, $c(|d|)$ is odd so by equation (2.1) we have $L(E_d, 1) \neq 0$. □

Remarks.
1- By the results of [Kol] and [BFH] or [Mu-Mu] we deduce that the rank of $E_d(\mathbb{Q})$ is 0 and that its Tate-Shafarevich group is finite.
2- It is a classical question to understand the ratio of d such that $L(E_d, 1) = 0$. Using random matrix theory and its link with L-functions, Conrey, Keating, Rubinstein and Snaith ([CKRS], [CKRS2]) have conjectured that if E is an elliptic curve over $\mathbb{Q}$, there exists a constant $c_E \geqslant 0$ such that:

$$\sum_{\substack{p \leqslant T \\ -p \text{ discriminant} \\ \varepsilon(E_{-p})=1 \\ L(E_{-p},1)=0}} 1 \sim c_E T^{3/4} \log(T)^{-5/8}$$

So, corollary 2 implies that the constant c_E can be 0.

Corollary 3 (under the Birch and Swinnerton-Dyer conjecture). *For all prime discriminants $d < 0$ such that $\varepsilon(E_d) = 1$ we have $2 \nmid |\text{Ш}(E_d)|$.*

Proof For such a discriminant, we already know that $L(E_d, 1) \neq 0$ and, in our example, the Birch and Swinnerton-Dyer conjecture predicts that:

$$|\text{Ш}(E_d)| = c(|d|)^2$$

Hence, $|\text{Ш}(E_d)|$ is odd. □

Remarks.
1- This seems to be in contradiction with the heuristic in [De1] which would have suggested a density of about 58% of $|\text{Ш}(E_d)|$ divisible by 2. Note that for odd primes p, the numerical data, performed by Rubinstein ([Rub]), about the density of $|\text{Ш}(E_d)|$ divisible by p are in close agreement with the predictions given by the heuristic (except for the p dividing $|E(\mathbb{Q})_{\text{tors}}|$). In fact, as we have seen, the effect of taking only prime discriminants d has a large consequence on the 2-divisibility of $S(E_d)$. This effect seems to disappear if we consider all discriminants $d < 0$ such that $\varepsilon(E_d) = 1$. For example, the density of the fundamental discriminants $-10^8 < d < 0$ such that $2 \mid S(E_d)$ is about 61.3%. We expect that the correct density is the one predicted by the heuristic but that the convergence is slow.
2- Using a 2-descent argument, it can be directly proved that the 2-parts of the Tate-Shafarevich groups $\text{Ш}(E_d)$ are all trivial and that the rank of E_d are all 0 whenever $d < 0$ runs through fundamental prime discriminants such that $\varepsilon(E_d) = 1$ ([Ant-Bun-Fre, exemple 1]). Hence, our results may also be seen as a check of the 2-part of the Birch and Swinnerton-Dyer conjecture for our family of quadratic twists.

3- Using also a 2-descent argument, one can obtain similar results for the case of "odd" quadratic twists of E by prime discriminants $d < 0$ ([De-Ro]): more precisely, if $d < 0$ is a prime discriminant such that $\varepsilon(E_d) = -1$ then we have $L'(E_d, 1) \neq 0$ and $2 \nmid |Ш(E_d)|$ (using a weak version of the Birch and Swinnerton-Dyer conjecture).

3 Generalization

Of course, our method can easily be adapted for many other examples (for instance $E =$"15a1", "21a1", "33a1"...). However, when the conductor N of the elliptic curve E is not prime, then the discriminants $d < 0$ should satisfy some more local conditions at the primes dividing N; indeed, if we want, for example, to apply the Kohnen-Zagier's theorem ([Koh-Zag]) for finding the weight 3/2 modular form, we must have, for all prime $\ell \mid N$, $\left(\frac{d}{\ell}\right) = \varepsilon_\ell$, where ε_ℓ is the eigenvalue of the Atkin-Lehner operator at ℓ. For instance, if we take $E = 15a1$, we prove, using the same technics as above, that for all prime discriminants $d < 0$ such that $\left(\frac{d}{3}\right) = 1$ and $\left(\frac{d}{5}\right) = -1$ then $L(E_d, 1) \neq 0$ and $S(E_d)$ is not divisible by 2.

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