Note on the frequency of vanishing of
$L$-functions of elliptic curves in a
family of quadratic twists

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Abstract

In this note, we give an example of an elliptic curve $E$ such that for all prime discriminants $d < 0$ for which the sign of the functional equation of the $L$-function of the quadratic twist $E_d$ of $E$ by $d$ is $+1$, we have $L(E_d, 1) \neq 0$. Furthermore, using the Birch and Swinnerton-Dyer conjecture, we prove that the Tate-Shafarevitch group of $E_d$, for all such $d$, has a trivial 2-part. Our method can be generalized to other examples.

1 Notations and introduction

Let $E$ be an elliptic curve defined over $\mathbb{Q}$ with conductor $N$ and with $L$-function $L(E, s) = \sum_n a(n)n^{-s}$. From the work of Wiles, Taylor ([Wil], [Tay-Wil]) and Breuil, Conrad, Diamond, Taylor ([Bre et al.]), $L(E, s)$ can be continued to the whole complex plane and satisfies a functional equation:

$$\Lambda(E, s) = \varepsilon(E)\Lambda(E, 2 - s),$$

where $\varepsilon(E) = \pm 1$ is the sign of the functional equation and $\Lambda(E, s)$ is given by:

$$\Lambda(E, s) = \left(\frac{\sqrt{N(E)}}{2\pi}\right)^s \Gamma(s)L(E, s).$$

Let $d$ be a fundamental discriminant, $\left(\frac{d}{\cdot}\right)$ its associated quadratic character and $E_d$ the quadratic twist of $E$ by $d$. Furthermore, we assume that $d$ is prime to $N$. Hence the conductor of $E_d$ is $Nd^2$, and the sign of the functional equation of $L(E_d, s) = \sum_n a(n) \left(\frac{d}{n}\right) n^{-s}$ is

$$\varepsilon(E_d) = \varepsilon(E) \left(\frac{d}{-N(E)}\right).$$

Classical questions are concerned with the distribution of the special values $L(E_d, 1)$ as $d$ runs through discriminants with $\varepsilon(E_d) = 1$. For example, one can ask for the density of those $d$ such that $L(E_d, 1) = 0$ or for the density
of those \(d\) such that \(p \mid |\Sha(E_d)|\), where \(p\) is a fixed prime and \(\Sha(E_d)\) is the Tate-Shafarevitch group\(^1\) of \(E_d\). The Birch and Swinnerton-Dyer conjecture gives a precise link between the value \(L(E_d, 1)\) and the order \(|\Sha(E_d)|\) of the Tate-Shafarevitch group.

Using elementary arithmetic on quadratic forms, we prove that if \(E\) is the elliptic curve “17a1” in Cremona’s table ([Cre]) then for all prime discriminants \(d < 0\) with \(\varepsilon(E_d) = 1\), we have \(L(E_d, 1) \neq 0\) and (using the Birch and Swinnerton-Dyer conjecture) \(2 \mid |\Sha(E_d)|\). Other examples can be handled with the same method.

2 The example

Throughout this section \(E\) denotes the elliptic curve with conductor \(N = 17\) defined by:

\[
E : y^2 + xy + y = x^3 - x^2 - x - 14.
\]

We consider the quadratic twists \(E_d\) of \(E\) by discriminants \(d < 0\) coprime with \(N\) such that \(\varepsilon(E_d) = 1\) (i.e. \(d \equiv 1, 2, 4, 8, 9, 13, 15, 16 \mod 17\)). By a theorem of Waldspurger ([Wal]), the values of \(L(E_d, 1)\) are related to the coefficients \(c(n)\) of a weight 3/2 modular form. More precisely, we have:

\[
L(E_d, 1) = \frac{\kappa}{\sqrt{|d|}} c(|d|)^2,
\]

(2.1)

where \(\kappa\) is a constant (here \(\kappa \approx 2.74573911\)) and where the modular form of weight 3/2, computed by Tornaria ([Tor]), is given by:

\[
\sum_n c(n)q^n = \frac{\theta_1(q) - \theta_2(q)}{2}
\]

with:

\[
\theta_1(q) = \sum_{(x,y,z)\in\mathbb{Z}^3} q^{3x^2+23y^2+23z^2-2xy-2xz-2yz}
\]

\[
\theta_2(q) = \sum_{(x,y,z)\in\mathbb{Z}^3} q^{7x^2+11y^2+20z^2-6xy-4xz-8yz}.
\]

We have:

**Theorem 1.** If \(d < 0\) is a prime discriminant with \(\varepsilon(E_d) = 1\), the coefficient \(c(-d)\) is odd.

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\(^1\)The Tate-Shafarevitch group of an elliptic curve \(E\) is some “cumbersome” group which, roughly speaking, measures the obstruction of a certain “local-global” principle (see [Sil] for a precise definition). It is conjectured that it is a finite group and, if so, one can prove that its order is a perfect square.
\textbf{Proof} Let $d$ be such a discriminant and $p = -d$. Remark that we have $p \equiv 3 \mod 4$ and that \((-17/p) = 1\). We let:

\[ Q_1(x, y, z) = 3x^2 + 23y^2 + 23z^2 - 2xy - 2xz - 22yz \]
\[ Q_2(x, y, z) = 7x^2 + 11y^2 + 20z^2 - 6xy - 4xz - 8yz. \]

We consider the following sets:

\[ R_1 = \{ (x, y, z) \in \mathbb{Z}^3, \ Q_1(x, y, z) = p \} \]
\[ R_2 = \{ (x, y, z) \in \mathbb{Z}^3, \ Q_2(x, y, z) = p \} \]

and we must prove that $|R_1| - |R_2| \equiv 2 \mod 4$. In fact, the two ternary quadratic forms $Q_1$ and $Q_2$ are invariant by the involutions $\iota : (x, y, z) \mapsto (-x, -y, -z)$ and $\tau : (x, y, z) \mapsto (x - z, y - z, -z)$. Hence, if $P = (x, y, z) \in R_i$ (for $i = 1, 2$), then $P, \iota(P), \tau(P)$ and $\iota \circ \tau(P)$ also belong to $R_i$ and these 4 points are distinct except if $z = 0$. Thus,

\[ |R_i| \equiv |\{(x, y) \in \mathbb{Z}^2, Q_i(x, y, 0) = p\}| \mod 4. \]

Hence for $|R_1| \mod 4$, we are led to study the number of solutions of

\[ p = 3x^2 - 2xy + 23y^2 = q_1(x, y) \]

and for $|R_2| \mod 4$ the number of solutions of

\[ p = 7x^2 - 6xy + 11y^2 = q_2(x, y). \]

There are 8 classes of primitive quadratic forms with discriminant $\Delta = -2^4 \times 17$ modulo $SL_2(\mathbb{Z})$. As a set of representatives we can choose the 8 following ones:

\[ q_1(x, y) = 3x^2 - 2xy + 23y^2 \quad q_2(x, y) = 7x^2 - 6xy + 11y^2 \]
\[ q_3(x, y) = 8x^2 - 4xy + 9y^2 \quad q_4(x, y) = 4x^2 + 17y^2 \]
\[ q_5(x, y) = x^2 + 68y^2 \]
\[ q_1(x, y) = 3x^2 + 2xy + 23y^2 \quad q_2(x, y) = 7x^2 + 6xy + 11y^2 \]
\[ q_3(x, y) = 8x^2 + 4xy + 9y^2 \quad q_4(x, y) = x^2 + 68y^2 \]
\[ q_5(x, y) = x^2 + 68y^2 \]
\[ q_2(x, y) = 7x^2 - 6xy + 11y^2 \]
\[ q_3(x, y) = 8x^2 - 4xy + 9y^2 \]
\[ q_4(x, y) = 4x^2 + 17y^2 \]
\[ q_5(x, y) = x^2 + 68y^2 \]
\[ q_6(x, y) = x^2 - 68y^2 \]
\[ q_7(x, y) = 8x^2 - 4xy + 9y^2 \]
\[ q_8(x, y) = 4x^2 + 17y^2 \]

Since we have $\left(\frac{-17}{p}\right) = 1$, the prime $p$ must be represented by one of these forms. Since $p \equiv 3 \mod 4$, the prime $p$ cannot be represented by the forms $q_3, q_3, q_4$ and $q_5$. Hence, we have two possibilities:

- The prime $p$ is represented by $q_1$ with only 2 solutions (and so it is for $\overline{q_1}$) and $p$ is not represented by $q_2$ (neither by $\overline{q_2}$).
- The prime $p$ is not represented by $q_1$ hence it is by $q_2$ with only 2 solutions (and so it is by $\overline{q_2}$).

In each case, we conclude that $|R_1| - |R_2| \equiv 2 \mod 4$ and so $c(p)$ is odd. \qed
Corollary 2. Let \( d < 0 \) be a prime discriminant such that \( \varepsilon(E_d) = 1 \), then we have \( L(E_d, 1) \neq 0 \).

**Proof** Indeed, \( c(|d|) \) is odd so by equation (2.1) we have \( L(E_d, 1) \neq 0 \). \( \square \)

Remarks.
1- By the results of [Kol] and [BFH] or [Mu-Mu] we deduce that the rank of \( E_d(\mathbb{Q}) \) is 0 and that its Tate-Shafarevich group is finite.
2- It is a classical question to understand the ratio of \( d \) such that \( L(E_d, 1) = 0 \). Using random matrix theory and its link with \( L \)-functions, Conrey, Keating, Rubinstein and Snaith ([CKRS], [CKRS2]) have conjectured that if \( E \) is an elliptic curve over \( \mathbb{Q} \), there exists a constant \( c_E \geq 0 \) such that:

\[
\sum_{p \leq T \atop -p \text{ discriminant} \atop \varepsilon(E_{-p})=1 \atop L(E_{-p},1)=0} 1 \sim c_E T^{3/4} \log(T)^{-5/8}
\]

So, corollary 2 implies that the constant \( c_E \) can be 0.

**Corollary 3 (under the Birch and Swinnerton-Dyer conjecture).** For all prime discriminants \( d < 0 \) such that \( \varepsilon(E_d) = 1 \) we have \( 2 \nmid |\text{III}(E_d)| \).

**Proof** For such a discriminant, we already know that \( L(E_d, 1) \neq 0 \) and, in our example, the Birch and Swinnerton-Dyer conjecture predicts that:

\[
|\text{III}(E_d)| = c(|d|)^2
\]

Hence, \( |\text{III}(E_d)| \) is odd. \( \square \)

Remarks.
1- This seems to be in contradiction with the heuristic in [De1] which would have suggested a density of about 58% of \( |\text{III}(E_d)| \) divisible by 2. Note that for odd primes \( p \), the numerical data, performed by Rubinstein ([Rub]), about the density of \( |\text{III}(E_d)| \) divisible by \( p \) are in close agreement with the predictions given by the heuristic (except for the \( p \) dividing \( |E(\mathbb{Q})_{\text{tors}}| \)). In fact, as we have seen, the effect of taking only prime discriminants \( d \) has a large consequence on the 2-divisibility of \( S(E_d) \). This effect seems to disappear if we consider all discriminants \( d < 0 \) such that \( \varepsilon(E_d) = 1 \). For example, the density of the fundamental discriminants \( -10^8 < d < 0 \) such that \( 2 \mid S(E_d) \) is about 61.3%. We expect that the correct density is the one predicted by the heuristic but that the convergence is slow.
2- Using a 2-descent argument, it can be directly proved that the 2-parts of the Tate-Shafarevich groups \( \text{III}(E_d) \) are all trivial and that the rank of \( E_d \) are all 0 whenever \( d < 0 \) runs through fundamental prime discriminants such that \( \varepsilon(E_d) = 1 \) ([Ant-Bun-Fre, exemple 1]). Hence, our results may also be seen as a check of the 2-part of the Birch and Swinnerton-Dyer conjecture for our family of quadratic twists.
3- Using also a 2-descent argument, one can obtain similar results for the case of “odd” quadratic twists of $E$ by prime discriminants $d < 0$ ([De-Ro]): more precisely, if $d < 0$ is a prime discriminant such that $\varepsilon(E_d) = -1$ then we have $L'(E_d, 1) \neq 0$ and $2 \nmid |\text{III}(E_d)|$ (using a weak version of the Birch and Swinnerton-Dyer conjecture).

3 Generalization

Of course, our method can easily be adapted for many other examples (for instance $E = “15a1”, “21a1”, “33a1”...$). However, when the conductor $N$ of the elliptic curve $E$ is not prime, then the discriminants $d < 0$ should satisfy some more local conditions at the primes dividing $N$; indeed, if we want, for example, to apply the Kohnen-Zagier’s theorem ([Koh-Zag]) for finding the weight 3/2 modular form, we must have, for all prime $\ell \mid N$, $(\frac{d}{\ell}) = \varepsilon_\ell$, where $\varepsilon_\ell$ is the eigenvalue of the Atkin-Lehner operator at $\ell$. For instance, if we take $E = 15a1$, we prove, using the same technics as above, that for all prime discriminants $d < 0$ such that $(\frac{d}{3}) = 1$ and $(\frac{d}{5}) = -1$ then $L(E_d, 1) \neq 0$ and $S(E_d)$ is not divisible by 2.

References


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