AVERAGES OF GROUPS INVOLVING p^ℓ-RANK AND COMBINATORIAL IDENTITIES

by

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Abstract. — We obtain averages of specific functions defined over (isomorphism classes) of some type of finite abelian groups. These averages are concerned with miscellaneous questions about the p^ℓ-ranks of these groups. We apply a classical heuristic principle to deduce from the averages precise predictions for the behavior of class groups of number fields and of Tate-Shafarevich groups of elliptic curves. Furthermore, the computations of these averages, which comes with an algebraic aspect, can also be reinterpreted with a combinatorial point of view. This allows us to recover and to obtain some combinatorial identities and to propose for them a natural algebraic context.

1. Introduction and notations

This article is concerned with the averages, in some sense, of functions defined over finite abelian groups and groups of type S. A finite abelian group G is called a group of type S if it is endowed with a non-degenerate, bilinear, alternating pairing:

$$\beta \ : \ G \times G \to \mathbb{Q}/\mathbb{Z}.$$

A group of type S has the form $G = H \times H$ where H is a finite abelian group. Reciprocally, each group G, of the form $H \times H$ has a unique structure of group of type S up to isomorphisms of such groups (i.e. isomorphisms preserving the underlying pairings). In particular, if G is a group of type S then $|\operatorname{Aut}^s(G)|$, the number of automorphisms of G that respect the (underlying) pairing β, does not depend on β (see, for example, [**Del01**] for these facts). Due to the above, the pairings associated to groups of type S will never be specified in the sequel.

For the rest of this paper, the letters H, H_1, H_2 are used for isomorphism classes of finite abelian groups and the letter G for isomorphism classes of groups of type S.

Let h be a complex-valued function defined on isomorphism classes of finite abelian groups and let $u \geqslant 0$. In [**CL84**], H. Cohen and H. Lenstra defined a certain u-average, $\mathrm{M}_u(h)$, of the function h (in particular, if h is the characteristic function of a property

$\mathcal{P}$, $\mathrm{M}_u(h)$ is called the u-probability of $\mathcal{P}$).
We write

$$\zeta(h,s)=\sum_{n\geqslant 1}\frac{1}{n^s}\sum_{H(n)}\frac{h(H)}{|\operatorname{Aut}(H)|},$$

where $\sum_{H(n)}$ means that the sum is over all isomorphism classes of finite abelian groups H of order n. It is proved in [**CL84**] that if h is non-negative, if $\zeta(h,s)$ converges for $\Re(s)>0$ and can be continued to a meromorphic function in $\Re(s)\geqslant 0$ with a unique pole of order at most 1 (the function $\zeta(h,s)$ could be holomorphic in $\Re(s)\geqslant 0$) at $s=0$ then

$$\mathrm{M}_u(h)=\lim_{s\to u}\frac{\zeta(h,s)}{\zeta(1,s)}.$$

H. Cohen and H. Lenstra introduced such averages as a heuristic model in order to give predictions for the behavior of the class groups of number fields varying in certain natural families. In particular, they proved that

$$\begin{aligned}\zeta(1,s) &= \sum_{n\geqslant 1}\frac{1}{n^s}\sum_{H(n)}\frac{1}{|\operatorname{Aut}(H)|}\\ &= \prod_{j\geqslant 1}\zeta(s+j)=\prod_{p\text{ prime}}\prod_{j\geqslant 1}\left(\frac{1}{1-\frac{1}{p^{s+j}}}\right),\end{aligned}$$

where $\zeta(\cdot)$ is the classical Riemann function. The reason of writing the above equality as an Euler product is that in general we compute u-averages of functions restricted to the p-parts of abelian groups and we recover global averages by taking the product of all the p-components.

The aim of [**Del01**] was to adapt the Cohen-Lenstra's model in order to give predictions for the behavior of Tate-Shafarevich groups of elliptic curves varying in certain natural families. Recall that, if they are finite (as it is classically conjectured), Tate-Shafarevich groups are groups of type S. One can define, for $u\geqslant 0$, the u-average (in the sense of groups of type S), $\mathrm{M}_u^s(g)$, for a function g defined on isomorphism classes of groups of type S. We let

$$\zeta^S(g,s)=\sum_{n\geqslant 1}\frac{1}{n^s}\sum_{G^S(n)}\frac{g(G)}{|\operatorname{Aut}^s(G)|}=\sum_{n\geqslant 1}\frac{1}{n^{2s}}\sum_{G^S(n^2)}\frac{g(G)}{|\operatorname{Aut}^s(G)|},$$

where $\sum_{G^S(n)}$ means that the sum is over all isomorphism classes of groups of type S, G of order n (the sum being empty if n is not a square). We have from [**Del01**], that if g is non-negative, if $\zeta^s(g,s-1)$ converges for $\Re(s)>0$ and can be continued to a meromorphic function in $\Re(s)\geqslant 0$ with a unique pole of order at most 1 (the function $\zeta^s(g,s-1)$ could be holomorphic in $\Re(s)\geqslant 0$) at $s=0$ then

$$\mathrm{M}_u^s(g)=\lim_{s\to u}\frac{\zeta^S(g,s-1)}{\zeta^S(1,s-1)}.^{(1)}$$

(1) The reason of taking "$s-1$" instead of s is explained in [**Del01**]. See also [**Del07**] for a precision of the heuristics for Tate-Shafarevich groups of elliptic curves of rank $\geqslant 0$.

It can be proved that

$$\begin{aligned}\zeta^{s}(1,s) &= \sum_{n\geqslant 1}\frac{1}{n^{s}}\sum_{G^{S}(n)}\frac{1}{|\operatorname{Aut}^{s}(G)|}\\ &= \prod_{j\geqslant 1}\zeta(2s+2j+1)=\prod_{p}\prod_{j\geqslant 1}\left(\frac{1}{1-\frac{1}{p^{2s+2j+1}}}\right).\end{aligned}$$

As for finite abelian groups, we will speak of the u-probability of $\mathcal{P}$ if g is the characteristic function of a property $\mathcal{P}$ of groups of type S. It will always be clear in the context whether we are concerned with average in the sense of finite abelian groups or in the sense of groups of type S.

Computations of u-averages can be viewed as algebraic interpretations of combinatorial identities. In that context, there is a generic tool that can help for going from u-averages of finite abelian groups to u-averages of groups of type S. Indeed, let p be a prime and assume that H and G are in fact p-groups (i.e. the order of H and G are a p-power). If $|H|=p^{n}$ and

$$H\simeq(\mathbb{Z}/p\mathbb{Z})^{\lambda_1}\oplus\left(\mathbb{Z}/p^{2}\mathbb{Z}\right)^{\lambda_2}\oplus\cdots\oplus(\mathbb{Z}/p^{s}\mathbb{Z})^{\lambda_s} \tag{1}$$

where $\lambda_1+2\lambda_2+\cdots+s\lambda_s=n$ is a partition of n, then, we denote $\mu_1,\mu_2,\ldots,\mu_s$ by

$$\begin{aligned}\mu_1 &= \lambda_1+\lambda_2+\cdots+\lambda_s;\\ \mu_2 &= \lambda_2+\cdots+\lambda_s;\\ &\vdots\\ \mu_s &= \lambda_s.\end{aligned}$$

Thus, we also have $\mu_1+\mu_2+\cdots+\mu_s=n$ and the number of automorphisms of H is given by (see for example [**Hal38**])

$$|\operatorname{Aut}(H)|=p^{\mu_1^2+\mu_2^2+\cdots+\mu_s^2}\prod_{1\leqslant j\leqslant s}\left(\frac{1}{p}\right)_{\lambda_j},$$

where $(q)_a$ is defined by

$$(q)_a=(1-q)(1-q^{2})\cdots(1-q^{a})\text{ with a}\in\mathbb{N}.$$

Now, if $G\simeq H\times H$ is a group of type S, we have

$$|\operatorname{Aut}^{s}(G)|=p^{2(\mu_1^2+\mu_2^2+\cdots+\mu_s^2)+n}\prod_{1\leqslant j\leqslant s}\left(\frac{1}{p^{2}}\right)_{\lambda_j}.$$

Whenever a certain equality involving $|\operatorname{Aut}(H)|$ holds, we can usually consider it as a formal identity in the variable $q=1/p$. To obtain the analogue for p-groups of type S (and involving $|\operatorname{Aut}^{s}(G)|$), one can apply, at convenient step, the following *rule*: first substitute p^2 for p and then multiply by p^n: we will use roughly this rule several times in this paper even if not explicitly mentioned.

The p^ℓ-rank, $r_{p^\ell}(H)$, of a finite abelian group H is the number of cyclic components of H of order divisible by p^ℓ. It is defined by

$$r_{p^\ell}(H) = \dim_{\mathbb{Z}/p\mathbb{Z}}(p^{\ell-1}H/p^\ell H).$$

With the decomposition (1), it is not difficult to see that

$$r_{p^\ell}(H) = \mu_\ell.$$

Furthermore, if $G \simeq H \times H$ is a group of type S then

$$r_{p^\ell}(G) = 2r_{p^\ell}(H) = 2\mu_\ell.$$

The first goal of this paper is to obtain some u-averages for finite abelian groups and for groups of type S. These u-averages are concerned with several questions related to p^ℓ-ranks. The first part deals with the u-average of the number of p^ℓ-torsion elements. We apply the heuristic principle to give predictions for the average number of p^ℓ-torsion elements in class groups of number fields and in Tate-Shafarevich groups of elliptic curves of rank 0 and 1 in natural families. These computations were motivated by questions of M. Bhargava (see the works of M. Bhargava and A. Shankar [**BS10b**], [**BS10a**]). In the second part, we study related questions about the probability laws of p^ℓ-ranks. This is done by generalizing and adapting a work of H. Cohen [**Coh85**]. Both in the first and second part, our calculations are, in fact, closed to some combinatorial questions and we investigate a little bit this observation. We recover, obtain or use some combinatorial identities and our work gives, in a way, a natural algebraic context for them.

2. Average number of p^ℓ-torsion elements

Let (λ) a partition of n. By this we mean that $(\lambda) = (\lambda_1, \lambda_2, \dots)$ and that we have $n = \lambda_1 + 2\lambda_2 + \cdots + s\lambda_s$ where $s = s(\lambda)$ is the largest integer with $\lambda_s \neq 0$. Then, we define $(\mu_j)_{1\leqslant j\leqslant s}$ by $\mu_j = \sum_{k=j}^{s} \lambda_k$. Let H be the finite abelian p-group of order p^n associated to this partition (λ):

$$H = (\mathbb{Z}/p\mathbb{Z})^{\lambda_1} \oplus \left(\mathbb{Z}/p^2\mathbb{Z}\right)^{\lambda_2} \oplus \cdots \oplus (\mathbb{Z}/p^s\mathbb{Z})^{\lambda_s}.$$

Let T_ℓ be the function defined by $T_\ell(H) = |H[p^\ell]| = |\{x \in H \colon p^\ell x = 0\}|$, so that $T_\ell(H)$ is the number of p^ℓ-torsion elements of H. We can easily see, with the notations above, that we have

$$T_\ell(H) = p^{\mu_1+\mu_2+\cdots+\mu_\ell} = p^{r_p(H)+r_{p^2}(H)+\cdots+r_{p^\ell}(H)}.$$

2.1. The classical weight. — We begin this part with the following lemma:

Lemma 1. — *Let ℓ be a positive integer with $\ell \leqslant n$, we have*

$$\sum_{H(p^n)} \frac{T_\ell(H) - T_{\ell-1}(H)}{|\operatorname{Aut}(H)|} = \frac{1}{p^{n-\ell}\,(1/p)_{n-\ell}}.$$

Proof. — The proof of the lemma is an adaptation of the proof of the Theorem 6.4 of [**CL84**]. For H_1 and H_2 two finite abelian groups, we denote by $\mathrm{Hom}_{\mathrm{inj}}(H_1, H_2)$ the subset of injective homomorphisms in $\mathrm{Hom}(H_1, H_2)$. If $\ell \leqslant n$, we have

$$\begin{aligned} H[p^\ell] &\simeq (\mathbb{Z}/p\mathbb{Z})^{\lambda_1} \oplus \cdots \oplus \left(\mathbb{Z}/p^\ell\mathbb{Z}\right)^{\lambda_\ell} \oplus \left(\mathbb{Z}/p^\ell\mathbb{Z}\right)^{\lambda_{\ell+1}} \oplus \cdots \oplus \left(\mathbb{Z}/p^\ell\mathbb{Z}\right)^{\lambda_s} \\ &\simeq \left((\mathbb{Z}/p\mathbb{Z})^{\lambda_1} \oplus \cdots \oplus \left(\mathbb{Z}/p^{\ell-1}\mathbb{Z}\right)^{\lambda_{\ell-1}} \right) \oplus \left(\mathbb{Z}/p^\ell\mathbb{Z}\right)^{\mu_\ell}. \end{aligned}$$

On the one hand, if we denote by $\mathrm{ord}(x)$ the order of a group element x, we have

$$\begin{aligned} |\mathrm{Hom}_{\mathrm{inj}}(\mathbb{Z}/p^\ell\mathbb{Z}, H[p^\ell])| &= |\{x \in \left(\mathbb{Z}/p^\ell\mathbb{Z}\right)^{\mu_\ell}, \mathrm{ord}(x) = p^\ell\}| \times p^{\lambda_1+\cdots+(\ell-1)\lambda_{\ell-1}} \\ &= p^{\ell\mu_\ell}\left(1 - \frac{1}{p^{\mu_\ell}}\right) \times p^{\lambda_1+2\lambda_2+\cdots+(\ell-1)\lambda_{\ell-1}} \\ &= p^{\mu_1+\cdots+\mu_\ell}\left(1 - \frac{1}{p^{\mu_\ell}}\right) \\ &= T_\ell(H) - T_{\ell-1}(H). \end{aligned}$$

The second equality comes from the fact that the number of elements of order p^ℓ in $(\mathbb{Z}/p^\ell\mathbb{Z})^b$ is $p^{b\ell}(1 - 1/p^b)$. On the other hand,

$$\begin{aligned} |\mathrm{Hom}_{\mathrm{inj}}(\mathbb{Z}/p^\ell\mathbb{Z}, H[p^\ell])| &= |\mathrm{Hom}_{\mathrm{inj}}(\mathbb{Z}/p^\ell\mathbb{Z}, H)| \\ &= |\mathrm{Aut}(\mathbb{Z}/p^\ell\mathbb{Z})| \cdot |\{H_1 \text{ subgroup of } H,\ H_1 \simeq \mathbb{Z}/p^\ell\mathbb{Z}\}|. \end{aligned}$$

Finally, Proposition 4.1 of [**CL84**] gives

$$|\mathrm{Aut}(\mathbb{Z}/p^\ell\mathbb{Z})| \sum_{H(p^n)} \frac{|\{H_1 \text{ subgroup of } H,\ H_1 \simeq \mathbb{Z}/p^\ell\mathbb{Z}\}|}{|\mathrm{Aut}(H)|} = \frac{1}{p^{n-\ell}\,(1/p)_{n-\ell}}.$$

□

We deduce from the lemma:

Theorem 2. — *For finite abelian groups, we have*

$$\sum_{n\geqslant 0} x^n \sum_{H(p^n)} \frac{T_\ell(H)}{|\mathrm{Aut}(H)|} = (1 - x^{\ell+1}) \prod_{j\geqslant 0} \frac{1}{1 - \frac{x}{p^j}} \qquad (|x| < 1),$$

and for groups of type S

$$\sum_{n\geqslant 0} x^{2n} \sum_{G^S(p^{2n})} \frac{T_\ell(G)}{|\mathrm{Aut}^s(G)|} = \left(1 - \frac{x^{2(\ell+1)}}{p^{\ell+1}}\right) \prod_{j\geqslant 0} \frac{1}{1 - \frac{x^2}{p^{2j+1}}} \qquad (|x| < 1).$$

Proof. — We prove the theorem by induction on ℓ. For $\ell = 0$, we have ([**CL84**])

$$\sum_{n\geqslant 0} x^n \sum_{H(p^n)} \frac{1}{|\mathrm{Aut}(H)|} = \sum_{n\geqslant 0} x^n \frac{(1/p)^n}{(1/p)_n},$$

and the results come from the following formal identity due to Euler:

$$\sum_{n\geqslant 0} \frac{q^n x^n}{(q)_n} = \prod_{j\geqslant 1} \frac{1}{1 - xq^j}.$$

Writing $T_\ell = (T_\ell - T_{\ell-1}) + T_{\ell-1}$ and using an induction argument, we have

$$\begin{aligned}\sum_{n\geqslant 0} x^n \sum_{H(p^n)} \frac{T_\ell(H)}{|\operatorname{Aut}(H)|} &= x^\ell \sum_{n\geqslant 0} x^n \frac{(1/p)^n}{(1/p)_n} - (1+x+\cdots+x^{\ell-1}) \prod_{j\geqslant 1} \frac{1}{1-\frac{x}{p^j}} \\ &= \prod_{j\geqslant 1} \frac{1}{1-\frac{x}{p^j}} (1+x+\cdots+x^\ell),\end{aligned}$$

which is exactly the result required for finite abelian groups. For groups of type S, we substitute p by p^2 and then x by x^2/p. ☐

We deduce from the theorem that we have

$$\begin{aligned}\zeta(T_\ell, s) &= \zeta(1,s)\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots+\frac{1}{p^{s\ell}}\right), \Re(s)>0, \\ \zeta^S(T_\ell, s) &= \zeta^S(1,s)\left(1+\frac{1}{p^{2s+1}}+\frac{1}{p^{2(2s+1)}}+\cdots+\frac{1}{p^{(2s+1)\ell}}\right), \Re(s)>-1,\end{aligned}$$

where $T_\ell(H)$ (resp. $T_\ell(G)$) is the number of p^ℓ-torsion elements in (the p-part of) H (resp. G). So,

Corollary 3. — *For finite abelian groups, the u-average of the number of p^ℓ-torsion elements is*

$$\mathrm{M}_u(T_\ell) = 1 + \frac{1}{p^u} + \cdots + \frac{1}{p^{u\ell}}.$$

For groups of type S, the u-average of the number of p^ℓ-torsion elements is

$$\mathrm{M}_u^S(T_\ell) = 1 + \frac{1}{p^{2u-1}} + \cdots + \frac{1}{p^{(2u-1)\ell}}.$$

We now apply the heuristic argument for Tate-Shafarevich groups of elliptic curves. Let $\mathcal{E}$ be some "natural" family of rank 0 elliptic curves E and suppose that the behavior of the p-part of $\text{Ш}(E)$ is not biased for this family. The average number of p^ℓ-torsion elements in $\text{Ш}(E)$ should be

$$1 + p + p^2 + \cdots + p^\ell.$$

Recently, M. Bhargava and A. Shankar ([**BS10b**], [**BS10a**]) obtained results on the Selmer groups of elliptic curves that give, in particular, very strong evidence towards the previous prediction for the family of all rank 0 elliptic curves. Our prediction also implies that the average number of elements in $\text{Ш}(E)$ of order exactly n should be n (since, by the heuristic principle, the p-parts should behave independently). In their work, M. Bhargava and A. Shankar obtained the same prediction with a totally different approach. One can also find in [**PR10**] some general theoretical results on Selmer groups.

For a family of rank 1 elliptic curves, the average should be

$$1 + \frac{1}{p} + \cdots + \frac{1}{p^\ell}. \tag{2}$$

Of course we can deduce similar conjectures for the p-part of class groups of quadratic fields for $p \neq 2$ (to our knowledge these predictions have not been done before). This implies that for a prime $p \geqslant 3$ the average number of p^ℓ-torsion elements in the class groups of imaginary quadratic fields should be $\ell+1$. In the case of real quadratic fields, the average number of p^ℓ-torsion elements in the class groups should be $1+\frac{1}{p}+\cdots+\frac{1}{p^\ell}$ exactly as in equation (2) (just due to the fact that $u = 2u-1$ if and only if $u = 1$).

As explained in the introduction, Theorem 2 can been seen as a formal identity (letting $q = 1/p$) and then we deduce the following corollary:

Corollary 4. — *We have*

$$\sum_{n\geqslant 0} x^n \sum_{(\lambda)=n} \frac{q^{\mu_1^2+\mu_2^2+\cdots+\mu_s^2-(\mu_1+\cdots+\mu_\ell)}}{(q)_{\lambda_1}\cdots(q)_{\lambda_s}} = \prod_{j\geqslant 1}\frac{1}{1-xq^j}\cdot(1+x+\cdots+x^\ell)$$

where $\sum_{(\lambda)=n}$ means that the sum is over all partitions (λ) of n.

Using the substitution $x \to xq$ and $\ell+1 \to \ell$, one can re-write the identity above as

$$\sum_{(\lambda)} \frac{x^{\mu_1+\mu_2+\cdots+\mu_s}q^{\mu_1^2+\mu_2^2+\cdots+\mu_s^2+\mu_\ell+\mu_{\ell+1}+\cdots+\mu_s}}{(q)_{\lambda_1}\cdots(q)_{\lambda_s}} = \prod_{j\geqslant 1}\frac{1}{1-xq^j}\cdot(1-(xq)^\ell) \tag{3}$$

where the sum is over all partitions λ (and where $s = s(\lambda)$ depends on the partition).

This equality should be compared with an identity due to Andrews ([**And74**]). Letting $k \geqslant 2$ and $1 \leqslant \ell \leqslant k$ be fixed integers, we define the series $Q_{q,k,\ell}(x)$ by

$$Q_{q,k,\ell}(x) = \sum_{n\geqslant 0} \frac{(-1)^n x^{kn} q^{\frac{(2k+1)n(n+1)}{2}-\ell n}(1-x^\ell q^{(2n+1)\ell})}{(q)_n\prod_{j\geqslant n+1}(1-xq^j)}.$$

In particular, for $x = 1$, Jacobi's identity gives

$$Q_{q,k,\ell}(1) = \prod_{\substack{n\geqslant 1\\ n\not\equiv 0,\pm\ell \mod 2k+1}} (1-q^n)^{-1}.$$

We can now state the results of Andrews ([**And74**, Theorem 1 and equation 2.5]):

***Theorem 5* (Andrews).** — *With the notations above:*

$$\sum_{\substack{(\lambda)\\ s(\lambda)\leqslant k-1}} \frac{x^{\mu_1+\mu_2+\cdots+\mu_s}q^{\mu_1^2+\mu_2^2+\cdots+\mu_s^2+\mu_\ell+\cdots+\mu_s}}{(q)_{\lambda_1}(q)_{\lambda_1}\cdots(q)_{\lambda_s}} = Q_{q,k,\ell}(x). \tag{4}$$

Equation (4) with $x = 1$ and the product formula are known as the full Andrews-Gordon identities. They are very famous and deep combinatorial generalizations of the Rogers-Ramanujan identities which are equation (4) for $k = 2$, $\ell = 1, 2$, $x = 1$ and the product formula (see also [**And98**]). We remark that through equation (3) we have in fact proved, with a completely algebraic aspect, Andrews's identity in the very special (and easier) case $k=\infty$ since

$$Q_{q,\infty,\ell}(x)=\prod_{j\geqslant 1}\frac{1}{1-xq^j}\cdot(1-(xq)^\ell).$$

Note that we could have directly obtained Theorem 2 using it. Nevertheless, we focus on the fact that our approach gives a natural algebraic interpretation of these identities.

2.2. The weight w_k. — This part is just concerned with the combinatorial aspects of u-averages and not with consequences on the the heuristics. In their article, Cohen and Lenstra defined in fact a more general (k,u)-average in which the weight $1/|\operatorname{Aut}(H)|$ has to be replaced by

$$w_k(H)=\frac{|\{\text{surjective homomorphisms } \mathbb{Z}^k\to H\}|}{|H|^k|\operatorname{Aut}H|}=\frac{\left(\frac{1}{p}\right)_k}{\left(\frac{1}{p}\right)_{k-r_p(H)}}\frac{1}{|\operatorname{Aut}G|},$$

with the convention that $(1/p)_a=0$ whenever $a<0$. Of course, one immediately sees that

$$\lim_{k\to+\infty}w_k(H)=\frac{1}{|\operatorname{Aut}(H)|}.$$

Duplicating the proof of Lemma 1, we obtain that for k,n two positive integers and for $\ell\in\mathbb{N}$ with $1\leqslant\ell\leqslant n$ we have

$$\sum_{H(p^n)}w_k(H)(T_\ell(H)-T_{\ell-1}(H))=\frac{\left(1-\frac{1}{p^k}\right)\left(\frac{1}{p}\right)_{n-\ell+k-1}p^{n-\ell}}{\left(\frac{1}{p}\right)_{k-1}\left(\frac{1}{p}\right)_{n-\ell}}.$$

Furthermore, for $\ell=1$:

$$\sum_{n\geqslant 0}x^n\sum_{H(p^n)}T_1(H)w_k(H)=\big(1+(1-p^{-k})x\big)\prod_{j=1}^{k}\frac{1}{1-\frac{x}{p^j}}.$$

Hence, an induction argument gives:

Proposition 6. — *For $k,\ell\geqslant 1$, we have*

$$\sum_{n\geqslant 0}x^n\sum_{H(p^n)}T_\ell(H)w_k(H)=\prod_{j=1}^{k}\frac{1}{1-\frac{x}{p^j}}\left(\frac{1-x^{\ell+1}-\frac{x}{p^k}+\frac{x^{\ell+1}}{p^k}}{1-x}\right).$$

From this formula we could easily recover the Dirichlet series that would be associated to T_ℓ and $w_k(H)$. Letting $q=1/p$, $x\leftarrow xq$ and substituting ℓ for $\ell+1$, we obtain:

Corollary 7. — *We have the following formal identity*

$$(5)\quad \sum_{\substack{(\lambda)\\ \mu_1\leqslant k}} \frac{(q)_k x^{\mu_1+\mu_2+\cdots+\mu_s}}{(q)_{k-\mu_1}\prod_{1\leqslant j\leqslant s}(q)_{\lambda_j}} q^{\mu_1^2+\mu_2^2+\cdots+\mu_s^2+\mu_\ell+\mu_{\ell+1}+\cdots+\mu_s} =$$

$$\prod_{j=0}^{k}\frac{1}{1-xq^{j+1}}\left(1-q^\ell x^\ell - xq^{k+1}+x^\ell q^{k+\ell}\right).$$

We recall that, in the formula above, $s = s(\lambda)$ depends on λ.

Of course, letting $k \to \infty$, we recover equation (3). Now, if we let $\ell \to \infty$ in equation (5) we obtain exactly one of Hall's formula ([**Hal38**]). Hence, equation (5) (which seems to be previously unknown) can be seen as one of its generalization. One could probably obtain equation (5) with some classical combinatorial tools, however, as before, our equation occurs naturally in an algebraic way. In an other direction, Stembridge ([**Ste90**]) gave some refinements of Hall's formula (mainly with $s(\lambda)$ bounded by a fixed m but still with $\ell = \infty$), which allowed him to recover Andrews' identity in the case $\ell = 1$ for all k (note that the cases $\ell = 1$ and $\ell = \infty$ are closely related), and therefore the Rogers-Ramanujan identities. Stembridge's method uses also some algebraic aspects but with a different approach. It should be interesting to obtain a Hall like formula with general ℓ through Stembridge's approach (i.e. with ℓ *and* $s(\lambda)$ bounded).

3. Probabilities on p^ℓ-ranks

The first aim of this section is to compute the probability laws of p^ℓ-ranks of finite abelian groups and groups of type S. More precisely, let $k \geqslant 1$, $\ell \geqslant 0$ be two integers. Having fixed $\ell+1$ non-negative integers $r_k \geqslant r_{k+1} \geqslant \cdots \geqslant r_{k+\ell}$ we are interested in the u-probability (and related questions) that a p-group H (resp. a p-group of type S, G) satisfies

$$r_{p^{\ell+j}}(H) = r_{k+j} \text{ (resp. } r_{p^{\ell+j}}(G) = 2r_{k+j}) \text{ for all } 0 \leqslant j \leqslant \ell.$$

We remark that the case $\ell = 0$ gives the probability laws for the p^k-rank and the case $k = 1$ gives the probability that a group has its p^j-ranks fixed for all $1 \leqslant j \leqslant \ell$. As this is a generalization and an adaptation of Theorem 2.4 of [**Coh85**] (which is for finite abelian groups and $\ell = 0$), we will just give the main steps of the proof. We also remark that the formula we are going to give for groups of type S cannot be obtained directly from [**Coh85**] and the rule given in the introduction. That is one of the reasons that we keep an indeterminate x all along the computations and need to use the full general Andrew's identity.

For convenience, we use the following notation: if H is a finite abelian group of order p^n, we write $n = e(H)$.

Suppose H is a p-group with $r_{p^{k+j}}(H) = r_{k+j}$ for $0 \leqslant j \leqslant \ell$. Then, we can write $H \simeq H_1 \times H_2$, with

$$H_1 \simeq \bigoplus_{1 \leqslant i \leqslant k+\ell-1} (\mathbb{Z}/p^i\mathbb{Z})^{\lambda_i}$$

where the λ_i are subject to certain conditions and

$$H_2 \simeq \bigoplus_{\ell \leqslant i \leqslant s} (\mathbb{Z}/p^i\mathbb{Z})^{\lambda_i}$$

for some s and $\lambda_\ell + \lambda_{\ell+1} + \cdots + \lambda_s = r_{k+\ell}$.

For H_1, we first remark that the conditions on the p^{k+j}-ranks mean that $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}$ are not restricted but that $\lambda_k, \ldots, \lambda_{k+\ell}$ have to be be fixed by the following ℓ equations:

$$\begin{array}{rcl} \lambda_k & = & r_k - r_{k+1}; \\ \lambda_{k+1} & = & r_{k+1} - r_{k+2}; \\ & \vdots & \\ \lambda_{k+\ell-1} & = & r_{k+\ell-1} - r_{k+\ell}. \end{array}$$

We let $(\widetilde{\lambda})$ be the restricted partition $(\widetilde{\lambda}) = (\lambda_1, \cdots, \lambda_{k-1})$ and

$$\begin{array}{rcl} \widetilde{\mu}_1 & = & \lambda_1 + \cdots + \lambda_{k-1}; \\ & \vdots & \\ \widetilde{\mu}_{k-1} & = & \lambda_{k-1}. \end{array}$$

Some technical calculations lead to the following equalities

$$e(H_1) = \sum_{j=1}^{k-1} \widetilde{\mu}_j + \sum_{j=k}^{k+\ell-1} r_j + (k-1)r_k - (k+\ell-1)r_{k+\ell}$$

and

$$\sum_{j=1}^{k+\ell-1} \widetilde{\mu}_j^2 = \sum_{j=1}^{k-1} \widetilde{\mu}_j^2 + 2(r_k - r_{k+\ell}) \sum_{j=1}^{k-1} \widetilde{\mu}_j + \\ (k-1)(r_k - r_{k+\ell})^2 + \sum_{j=k}^{k+\ell-1} r_j^2 - 2r_{k+\ell} \sum_{j=k}^{k+\ell-1} r_j + \ell r_{k+\ell}^2.$$

Furthermore, one can see that

$$|\operatorname{Aut}(H)| = |\operatorname{Aut}(H_1)||\operatorname{Aut}(H_2)|p^{2re(H_1)}.$$

We now fix an integer $1 \leqslant a \leqslant k$ and we remark that

$$|H/p^{a-1}H| = p^{r_k(a-1)} p^{\widetilde{\mu}_1 + \widetilde{\mu}_2 + \cdots + \widetilde{\mu}_{k-1} - (\widetilde{\mu}_a + \widetilde{\mu}_{a+1} + \cdots + \widetilde{\mu}_{k-1})}.$$

Finally, we can state, with the convention that $Q_{q,1,\ell}(x) = 1$:

Theorem 8. — *Let $k \geqslant 1$, $r_k \geqslant r_{k+1} \geqslant \cdots \geqslant r_{k+\ell}$ be $\ell+1$ non-negative integers and $1 \leqslant a \leqslant k$.*
For finite abelian groups, we have

$$\sum_{n\geqslant 0} x^n \sum_{\substack{H(p^n)\\ r_{p^k}(H)=r_k\\ \cdots \\ r_{p^{k+\ell}}(H)=r_{k+\ell}}} \frac{|H/p^{a-1}H|}{|\operatorname{Aut} H|} =$$

$$p^{r_k(a-1)} \frac{x^{kr_k+r_{k+1}+\cdots+r_{k+\ell}} \displaystyle\prod_{j=1}^{r_{k+\ell}} \left(1-\frac{x}{p^j}\right)^{-1} Q_{\frac{1}{p},k,a}\left(\frac{x}{p^{2r_k-1}}\right)}{p^{kr_k^2+r_{k+1}^2+\cdots+r_{k+\ell}^2} \left(\frac{1}{p}\right)_{r_{k+\ell}} \displaystyle\prod_{j=k}^{k+\ell-1} \left(\frac{1}{p}\right)_{r_j-r_{j+1}}}.$$

For groups of type S, we have

$$\sum_{n\geqslant 0} x^{2n} \sum_{\substack{G^S(n^2)\\ r_{p^k}(G)=2r_k\\ \cdots \\ r_{p^{k+\ell}}(G)=2r_{k+\ell}}} \frac{|G/p^{a-1}G|}{|\operatorname{Aut}^s G|} =$$

$$p^{2r_k(a-1)} \frac{x^{2(kr_k+r_{k+1}+\cdots+r_{k+\ell})} \displaystyle\prod_{j=1}^{r_{k+\ell}} \left(1-\frac{x^2}{p^{2j+1}}\right)^{-1} Q_{\frac{1}{p^2},k,a}\left(\frac{x^2}{p^{4r_k-1}}\right)}{p^{2(kr_k^2+r_{k+1}^2+\cdots+r_{k+\ell}^2)} \left(\frac{1}{p^2}\right)_{r_{k+\ell}} \displaystyle\prod_{j=k}^{k+\ell-1} \left(\frac{1}{p^2}\right)_{r_j-r_{j+1}}}.$$

Proof. — We let

$$(*) := \sum_{\substack{H\\ \text{as above}}} \frac{|H/p^{a-1}H| x^{e(H)}}{|\operatorname{Aut} H|}.$$

We have

$$(*) \quad = \quad p^{r_k(a-1)} \sum_{\substack{H_1,\, H_2\\ \text{as above}}} \left(\frac{x^{e(H_1)} x^{e(H_2)} p^{-2re(H_1)}}{|\operatorname{Aut}(H_1)||\operatorname{Aut}(H_2)|} p^{\widetilde{\mu}_1+\cdots+\widetilde{\mu}_{k-1}-(\widetilde{\mu}_a+\cdots+\widetilde{\mu}_{k-1})} \right).$$

So,

$$\begin{aligned}(*) \quad &= \quad \frac{p^{r_k(a-1)}}{\left(\frac{1}{p}\right)_{r_{k+\ell}}} \sum_{H_1} \frac{p^{-2re(H_1)} x^{e(H_1)} p^{\widetilde{\mu}_1+\cdots+\widetilde{\mu}_{k-1}-(\widetilde{\mu}_a+\cdots+\widetilde{\mu}_{k-1})}}{|\operatorname{Aut}(H_1)|} \sum_{\substack{J\subset (p^{\ell}\mathbb{Z})^r\\ H_2\simeq \mathbb{Z}^r/J}} \frac{x^{e(H_2)}}{|H_2|^r} \\ &= \quad \frac{p^{r_k(a-1)}}{(1/p)_{r_{k+\ell}}} ST.\end{aligned}$$

We have used the fact (see [**Coh85**]) that for a given H_2, the number of $J \subset (p^\ell\mathbb{Z})^r$ such that $H_2 \simeq \mathbb{Z}^r/J$ is equal to $|H_2|^r(1/p)_r/|\operatorname{Aut}(H_2)|$, and we denoted

$$\begin{aligned} S &= \sum_{\substack{H_1 \\ \text{as above}}} \frac{p^{-2re(H_1)}x^{e(H_1)}}{|\operatorname{Aut}(H_1)|} p^{\widetilde{\mu}_1+\cdots+\widetilde{\mu}_{k-1}-(\widetilde{\mu}_a+\cdots+\widetilde{\mu}_{k-1})}; \\ T &= \sum_{\substack{J\subset(p^\ell\mathbb{Z})^r \\ H_2\simeq\mathbb{Z}^r/J}} \frac{x^{e(H_2)}}{|H_2|^r}. \end{aligned}$$

Still following [**Coh85**], we have for T:

$$\begin{aligned} T &= \sum_{\alpha\geqslant 0} p^{-(\ell r+\alpha)r}x^{\ell r+\alpha} \sum_{\substack{J\subset(p^\ell\mathbb{Z})^r \\ |(p^\ell\mathbb{Z})^r/J|=p^\alpha}} 1 \\ &= p^{-\ell r^2}x^{\ell r}\sum_{\alpha\geqslant 0} x^\alpha \left(\frac{1}{p}\right)^\alpha \frac{(1/p)_{\alpha+r-1}}{(1/p)_{r-1}(1/p)_\alpha} \\ &= (1/p)^{\ell r^2}\, x^{\ell r} \prod_{j=1}^{r} \frac{1}{1-\frac{x}{p^j}}. \end{aligned}$$

Finally, we obtain

$$\begin{aligned} (*) = \frac{p^{r_k(a-1)}}{(1/p)_{r_{k+\ell}}} \frac{x^{(k-1)r_k+r_k+r_{k+1}+\cdots+r_{k+\ell}}\prod_{j=1}^{r_{k+\ell}}(1-xp^{-j})^{-1}}{\prod_{j=k}^{k+\ell-1}(1/p)_{r_j-r_{j+1}}p^{(k-1)r_k^2+r_k^2+r_{k+1}^2+\cdots+r_{k+\ell}^2}} \\ \times \sum_{\lambda_1,\cdots,\lambda_{k-1}} \left(\frac{p^{-2r_k(\widetilde{\mu}_1+\cdots+\widetilde{\mu}_{k-1})}x^{\widetilde{\mu}_1+\cdots+\widetilde{\mu}_{k-1}}}{p^{\widetilde{\mu}_1^2+\cdots+\widetilde{\mu}_{k-1}^2}\prod_{j=1}^{k-1}(1/p)_{\lambda_j}} p^{\widetilde{\mu}_1+\cdots+\widetilde{\mu}_{k-1}-(\widetilde{\mu}_a+\cdots+\widetilde{\mu}_{k-1})} \right), \end{aligned}$$

and a little calculation gives Theorem 8. □

We are now able to give two direct applications of Theorem 8 related to the computations of averages and to conjectures on class groups and on Tate-Shafarevich groups (via the heuristic principle).

3.1. The case $\ell = 0$. — This is the original goal of [**Coh85**] for finite abelian groups. On the one hand, we recover Theorem 2.4 of [**Coh85**] (with a slightly different notation):

$$\sum_{n\geqslant 1}\frac{1}{n^s}\sum_{\substack{H(n)\\ r_{p^k}=r}} \frac{|H/p^{a-1}H|}{|\operatorname{Aut} H|} = \frac{\prod_{j\geqslant r+1}\left(1-\frac{1}{p^{j+s}}\right)}{\left(\frac{1}{p}\right)_r p^{kr(s+r)-r(a-1)}} Q_{\frac{1}{p},k,a}\left(\frac{1}{p^{2r+s-1}}\right)\zeta(1,s).$$

If we let $a = 1$, then we obtain that the u-probability that a finite abelian group has p^k-rank equal to r is

$$\frac{\prod_{j\geqslant r+1}\left(1-\frac{1}{p^{j+u}}\right)}{\left(\frac{1}{p}\right)_r p^{kr(u+r)}} Q_{\frac{1}{p},k,1}\left(\frac{1}{p^{2r+u-1}}\right). \tag{6}$$

Note that $Q_{\frac{1}{p},k,1}(p) = Q_{\frac{1}{p},k,k}(1)$, which allows us to recover the 0 and 1-probabilities as a product formula as in [**Coh85**, Corollary 3.1] (and in fact, with our parameter x, the formula with $a > 1$ is useless for the 0 and 1-probability we have in mind).

On the other hand, we have for groups of type S (for simplicity we only give the Dirichlet function for $a = 1$)

$$\sum_{n\geqslant 1}\frac{1}{n^{2s}}\sum_{\substack{G^S(n^2)\\ r_{p^k}=2r}}\frac{1}{|\operatorname{Aut}^s G|} = \frac{\prod_{j\geqslant r+1}\left(1-\frac{1}{p^{2s+2j+1}}\right)}{\left(\frac{1}{p^2}\right)_r p^{kr(2s+2r+1)}} Q_{\frac{1}{p^2},k,1}\left(\frac{1}{p^{4r+2s-1}}\right)\zeta^S(1,s).$$

From the formula, we deduce:

Proposition 9. — *For groups of type S, the u-probability that a group of type S has p^k-rank equal to $2r$ is*

$$\frac{\prod_{j\geqslant r+1}\left(1-\frac{1}{p^{2u+2j-1}}\right)}{\left(\frac{1}{p^2}\right)_r p^{kr(2u+2r-1)}} Q_{\frac{1}{p^2},k,1}\left(\frac{1}{p^{4r+2u-3}}\right).$$

Once again, we apply the heuristic argument for Tate-Shafaverich groups of elliptic curves. Let $\mathcal{E}$ be a "natural" family of rank 0 elliptic curves E such that the behavior of the p-part of $\text{Ш}(E)$ is not biased. The probability that the p^k-rank of $\text{Ш}(E)$ is equal to $2r$ should be

$$\mathcal{P}_0(k,r) = \frac{\prod_{j\geqslant r+1}\left(1-\frac{1}{p^{2j-1}}\right)}{\left(\frac{1}{p^2}\right)_r p^{kr(2r-1)}} Q_{\frac{1}{p^2},k,1}\left(\frac{1}{p^{4r-3}}\right).$$

For a family of rank 1 elliptic curves, the probability should be

$$\mathcal{P}_1(k,r) = \frac{\prod_{j\geqslant r+1}\left(1-\frac{1}{p^{2j+1}}\right)}{\left(\frac{1}{p^2}\right)_r p^{kr(2r+1)}} Q_{\frac{1}{p^2},k,1}\left(\frac{1}{p^{4r-1}}\right).$$

The form of the two formulas above (in particular the fact that the powers of p in the index and in the argument of Q have different parities) suggests that it is probably hopeless to find a product formula for the u-probabilities related to Tate-Shafarevich groups (unlike the case of class groups of quadratic fields). Here are few numerical approximations of values of $\mathcal{P}_0(k,r)$ and $\mathcal{P}_1(k,r)$.

$\mathcal{P}_i(k,r)$	$p=2$	$p=3$	$p=5$
$\mathcal{P}_0(2,0)$	0.70837	0.87962	0.95866
$\mathcal{P}_0(2,1)$	0.29128	0.12036	0.04133
$\mathcal{P}_0(2,2)$	0.00033	10^{-6}	10^{-9}
$\mathcal{P}_0(3,0)$	0.85416	0.95987	0.99173
$\mathcal{P}_0(3,1)$	0.14582	0.04012	0.00826
$\mathcal{P}_1(2,0)$	0.97981	0.99846	0.99993
$\mathcal{P}_1(2,1)$	0.02018	0.00153	10^{-5}
$\mathcal{P}_1(3,0)$	0.99747	0.99994	0.99999

3.2. The case $k=1$. — We must have $a=1$. Let us fix ℓ integers $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_{\ell-1} \geqslant r_\ell \geqslant 0$. Then Theorem 8 gives

$$\sum_{\substack{H \\ r_p(H)=r_1 \\ \cdots \\ r_{p^\ell}(H)=r_\ell}} \frac{x^{e(H)}}{|\operatorname{Aut} H|} = \frac{x^{r_1+\cdots+r_\ell}}{p^{r_1^2+\cdots+r_\ell^2}\left(\frac{1}{p}\right)_{r_\ell}\prod_{j=1}^{\ell-1}\left(\frac{1}{p}\right)_{r_j-r_{j+1}}} \prod_{j=1}^{r}\frac{1}{1-\frac{x}{p^j}}. \tag{7}$$

It is clear that the factor $x^{r_1+\cdots+r_\ell}p^{-r_1^2-\cdots-r_\ell^2}/\prod_{j=1}^{\ell-1}(1/p)_{r_j-r_{j+1}}$ appears in each term of the sum in the left hand side. Naturally, this factor also appears in the right hand side.

Corollary 10. — *We have for finite abelian groups*

$$\sum_{n\geqslant 1}\frac{1}{n^s}\sum_{\substack{H(n) \\ r_p(H)=r_1 \\ \cdots \\ r_{p^\ell}(H)=r_\ell}}\frac{1}{|\operatorname{Aut} H|} = \frac{\zeta(1,s)\prod_{j\geqslant r_\ell+1}\left(1-\frac{1}{p^{s+j}}\right)}{p^{r_1^2+\cdots+r_\ell^2+s(r_1+\cdots+r_\ell)}\left(\frac{1}{p}\right)_{r_\ell}\prod_{j=1}^{\ell-1}\left(\frac{1}{p}\right)_{r_j-r_{j+1}}},$$

and for groups of type S

$$\sum_{n\geqslant 1}\frac{1}{n^{2s}}\sum_{\substack{G^S(n^2) \\ r_p(G)=2r_1 \\ \cdots \\ r_{p^\ell}(G)=2r_\ell}}\frac{1}{|\operatorname{Aut}^s G|} = \frac{\zeta^S(1,s)\prod_{j\geqslant r_\ell+1}\left(1-\frac{1}{p^{2s+2j+1}}\right)}{p^{2(r_1^2+\cdots+r_\ell^2)+(2s+1)(r_1+\cdots+r_\ell)}\left(\frac{1}{p^2}\right)_{r_\ell}\prod_{j=1}^{\ell-1}\left(\frac{1}{p^2}\right)_{r_j-r_{j+1}}}.$$

We deduce from the previous the u-probability laws:

Corollary 11. — *Let $\ell \geqslant 1$ and $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_{\ell-1} \geqslant r_\ell \geqslant 0$. For finite abelian groups, the u-probability that a finite abelian group has p^j-rank equal to r_j for all $1 \leqslant j \leqslant \ell$ is*

$$\frac{\displaystyle\prod_{j\geqslant r_\ell+1}\left(1-\frac{1}{p^{u+j}}\right)}{p^{r_1^2+\cdots+r_\ell^2+u(r_1+\cdots+r_\ell)}\left(\frac{1}{p}\right)_{r_\ell}\displaystyle\prod_{j=1}^{\ell-1}\left(\frac{1}{p}\right)_{r_j-r_{j+1}}}.$$

For groups of type S, the u-probability that a group of type S has p^j-rank equal to $2r_j$ for all $1 \leqslant j \leqslant \ell$ is

$$\frac{\displaystyle\prod_{j\geqslant r_\ell+1}\left(1-\frac{1}{p^{2u+2j-1}}\right)}{p^{2(r_1^2+\cdots+r_\ell^2)+(2u-1)(r_1+\cdots+r_\ell)}\left(\frac{1}{p^2}\right)_{r_\ell}\displaystyle\prod_{j=1}^{\ell-1}\left(\frac{1}{p^2}\right)_{r_j-r_{j+1}}}.$$

Once again, we could deduce, from the heuristic principle, conjectures concerning class groups and Tate-Shafarevich groups.

3.3. Examples and identities. — We obtain from the first and second parts some combinatorial identities (and we only focus on this aspect). The following formula seems to be previously unknown:

Corollary 12. — *We have*

$$\sum_{0\leqslant r_\ell\leqslant\cdots\leqslant r_1}\frac{q^{r_1^2+\cdots+r_\ell^2+(u-1)(r_1+\cdots+r_\ell)}\displaystyle\prod_{j\geqslant r_\ell+1}\left(1-q^{u+j}\right)}{(q)_{r_\ell}\displaystyle\prod_{j=1}^{\ell-1}(q)_{r_j-r_{j+1}}}=1+q^u+\cdots+q^{u\ell}.$$

Proof. — We apply

$$\sum_{0\leqslant r_\ell\leqslant\cdots\leqslant r_1}u\text{-prob}(r_p(H)=r_1,r_{p^2}(H)=r_2,\cdots r_{p^\ell}(H)=r_\ell)T_\ell(H)=\mathrm{M}_u(T_\ell).$$

Then, we use Corollary 3 and Corollary 11 and substitue $1/p \to q$. □

In particular, if u is an integer, the formula in Corollary 12 becomes

$$\sum_{0\leqslant r_\ell\leqslant\cdots\leqslant r_1}\frac{q^{r_1^2+\cdots+r_\ell^2+(u-1)(r_1+\cdots+r_\ell)}}{(q)_{r_\ell+u}\,(q)_{r_\ell}\displaystyle\prod_{j=1}^{\ell-1}(q)_{r_j-r_{j+1}}}=\frac{1+q^u+\cdots+q^{u\ell}}{(q)_\infty}. \tag{8}$$

The formula (8) and the formula in Corollary 12 are nearly of the same type as the ones in [**Hal38**] and in [**Ste90**]. They do not seem however to be direct consequences of it. There are neither special cases of the generalizations of Stembridge's formulas given in [**IJZ06**] and in [**JZ05**].

Finally, we conclude with a last remark. Let $k \geqslant 1$ and $r_k \geqslant r_{k+1} \geqslant \cdots \geqslant r_{k+\ell}$ be $\ell+1$ non-negative integers. Theorem 8 can be re-written as

$$(9)\qquad \sum_{\substack{(\lambda)\\ \mu_k=r_k\\ \mu_{k+1}=r_{k+1}\\ \cdots\\ \mu_{k+\ell}=r_{k+\ell}}} \frac{x^{\mu_1+\mu_2+\cdots+\mu_s}q^{\mu_1^2+\mu_2^2+\cdots+\mu_s^2}}{(q)_{\lambda_1}(q)_{\lambda_2}\cdots(q)_{\lambda_s}} q^{-\mu_1-\mu_2-\cdots-\mu_s+\mu_a+\mu_{a+1}+\cdots+\mu_s} =$$

$$q^{(1-a)r_k}\frac{x^{kr_k+r_{k+1}+\cdots+r_{k+\ell}}q^{kr_k^2+r_{k+1}^2+\cdots+r_{k+\ell}^2}Q_{q,\ell,1}(xq^{2r_k-1})}{\prod_{j=1}^{r_{k+\ell}}(1-xq^j)(q)_{r_{k+\ell}}\prod_{j=k}^{k+\ell-1}(q)_{r_j-r_{j+1}}}.$$

We fix $r_{k+\ell}=r$ and write

$$\sum_{\substack{(\lambda)\\ \mu_{k+\ell}=r}} X = \sum_{\substack{r_k,r_{k+1},\cdots r_{k+\ell-1}\\ r_k\geqslant r_{k+1}\cdots\geqslant r_{k+\ell-1}\geqslant r}} \sum_{\substack{(\lambda)\\ \mu_k=r_k\\ \cdots\\ \mu_{k+\ell-1}=r_{k+\ell-1}\\ \mu_{k+\ell}=r}} X,$$

and we apply equation (9) with convenient parameters to the left hand side and to the inner sum of right hand side (in fact, we also take $r=0$ since the general case $r>0$ is a direct consequence of it). Then, we use the substitution $x \to x/q$, we reindex r_k, r_{k+1}, $\cdots$, $r_{k+\ell-1}$ by r_1, r_2, $\cdots$, r_ℓ and we obtain directly the following general recursion formula on the $Q_{q,k,a}$:

Proposition 13. — *We have for $\ell, k \geqslant 1$, $r \geqslant 0$ and $1 \leqslant a \leqslant k$*

$$(10)\quad Q_{q,k+\ell,a}(x) = \sum_{r_1\geqslant\cdots\geqslant r_\ell\geqslant 0}\left(x^{kr_1+r_2+\cdots+r_\ell}q^{kr_1^2+r_2^2+\cdots+r_\ell^2+kr_1+r_2+\cdots+r_\ell}\frac{q^{(1-a)r_1}Q_{q,k,a}(xq^{2r_1})}{(q)_{r_\ell}\prod_{j=1}^{\ell-1}(q)_{r_j-r_{j+1}}}\right).$$

When $\ell=1$, formula (10) is exactly one of the inputs needed by Andrews for proving equation (4). Formula (10) is probably also a consequence of consecutive applications of the special case $\ell=1$. The questions of using or obtaining other classical combinatorial identities with this algebraic aspect will be discussed elsewhere.

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